Strategic Reasoning and Robust Predictions:

Putting Epistemic Game Theory to Work

Pierpaolo *Battigalli* and Marciano Siniscalchi

CMID 2020


Jif
PB&J
Strawberry
Kellogg's
Krave
CHOCOLATE
Kellogg's
Krave
DOUBLE CHOCOLATE
Kellogg's
Krave
S'MORES
Raisin Nut Bran
Chex
Chex
Chex
Cheerios
Cheerios
KiX
KiX
Cheerios
INSTANTLY WIN GOLD $100
Kellogg's
CORN POPS
WIN GOLD $100
Kellogg's
CORN POPS
Kellogg's
APPLE JACKS
Kellogg's
APPLE JACKS
FINDING DORY
2/$5
2/$5
Toasted Oats

Epistemic Game Theory in two slogans:

Take Strategic Uncertainty Seriously

Explicit is Better than Implicit

Today's Agenda

Applications of EGT

Focus on examples

- First-price auctions
- Robust implementation

(but first, a crash course)

Payoff Uncertainty: "Textbook" Type Structures

$(T_i, \beta_i)_{i \in I}$:

- T_i : set of types
- $U_i(t_i, t_{-i}, s_i, s_{-i})$: expected utility payoff (in strategic form)
- $\beta_i : T_i \to \Delta(T_{-i})$: belief map
- $\beta_i(\cdot)$ usually generated by common prior

Types do double duty: information about payoffs, parameterization of exogenous interactive beliefs

Payoff Uncertainty: Harsanyi's Actual Type Structures

$(\Theta_i, \bar{T}_i, \bar{\beta}_i)_{i \in I}$:

- Θ_i: i's private information about payoffs
- $u_i(\theta_i, \theta_{-i}, s_i, s_{-i})$: utility payoff (in strat. form)
- $\bar{T}_i$: set of exogenous-belief types
- $\bar{\beta}_i : \bar{T}_i \to \Delta(\Theta_{-i} \times \bar{T}_{-i})$: exogenous-belief map
- common prior is optional

Adding Strategic Uncertainty: Epistemic Type Structures

$(\Theta_i, S_i, T_i, \beta_i)_{i\in I}$:

- Θ_i: i's private information about payoffs
- $u_i(\theta_i, \theta_{-i}, s_i, s_{-i})$: utility payoff (in strat. form)
- T_i : set of endogenous-belief types
- $\beta_i : T_i \to \Delta(\Theta_{-i} \times S_{-i} \times T_{-i})$: endo-belief map
- common prior about Θ optional
- in *dynamic* games, beliefs=conditional probability systems (CPS) $\beta_i(t_i) \in \Delta^{\mathcal{H}}(\Theta_{-i} \times S_{-i} \times T_{-i})$, with $\beta_{i,h}(t_i) \in \Delta(\Theta_{-i} \times S_{-i}(h) \times T_{-i})$ local belief, and $S_{-i}(h)$ =strategies allowing h

The Dividend: Rationality and Belief as Events

Rationality:

$$R_i = \left\{ (\theta_i, s_i, t_i) : s_i \text{ (seq)BR to } \mathrm{marg}_{\Theta_{-i} \times S_{-i}} \beta_i(t_i) \right\}$$

(in *dynamic* games, sequential best reply to marginal CPS)

Belief: for $E_{-i} \subseteq \Theta_{-i} \times S_{-i} \times T_{-i}$,

$$\mathrm{B}_i(E_{-i}) = \{(\theta_i, s_i, t_i) : \beta_i(t_i)(E_{-i}) = 1\}$$

For *dynamic* games, local belief at history h:

$$\mathrm{B}_{i,h}(E_{-i}) = \{(\theta_i, s_i, t_i) : \beta_{i,h}(t_i)(E_{-i}) = 1\}$$

Strong belief (Battigalli & Siniscalchi JET 2002):

$$\mathrm{SB}_i(E_{-i}) = \bigcap_{h : E_{-i} \cap [\Theta_{-i} \times S_{-i}(h) \times T_{-i}] \neq \emptyset} \mathrm{B}_{i,h}(E_{-i})$$

Belief whenever possible

Epistemic Conditions

Start with *static* games:

RCBR: Rationality and common belief thereof

$$R_i^1 = R_i; \qquad R_i^k = R_i^{k-1} \cap \mathrm{B}_i(R_{-i}^{k-1}) \quad (k > 1)$$

A.k.a. "belief-free." Better: free of restrictions on interactive beliefs about Θ

Directed RCBR: for $\Delta_{i,\theta_i} \subseteq \Delta(\Theta_{-i} \times S_{-i})$, (transparent) restrictions on first-order beliefs

$$[\Delta_i] = \{(\theta_i, s_i, t_i) : \mathrm{marg}_{\Theta_{-i} \times S_{-i}} \beta_i(t_i) \in \Delta_{i,\theta_i}\};$$

then, as above,

$$R_i^{\Delta,1} = R_i \cap [\Delta_i]; \quad R_i^{\Delta,k} = R_i^{\Delta,k-1} \cap \mathrm{B}_i(R_{-i}^{\Delta,k-1}) \quad (k > 1)$$

For *dynamic* games, replace belief $\mathrm{B}_i(\cdot)$ with strong belief $\mathrm{SB}_i(\cdot)$ and RCBR with RCSBR (best rationalization principle)
Battigalli & Siniscalchi JET 1999, 2002, BEJTE 2003

Solution Concepts as Algorithms

Solution concepts help make EGT operational

Characterization of behavioral implications

Iterative deletion of (qualified) non-best responses for each θ_i

Relevant today (Battigalli RiE 2003, Battigalli & Siniscalchi BEJTE 2003):

- RC(S)BR; s_i is ("belief-free") rationalizable for θ_i: k-th round is $S_i^k(\theta_i)$ (different from rationalizability in Bayesian games, see on this Battigalli, Di Tillio, Grillo, & Penta BEJTE 2011)
- Δ-RC(S)BR; s_i is Δ-rationalizable for θ_i: k-th round is $S_i^{\Delta,k}(\theta_i)$

(will illustrate details in examples)

EGT and Bayesian Games (1)

Static game with payoff uncertainty:

$$G = (\Theta_i, S_i, u_i : \Theta \times S \to \mathbb{R})_{i \in I}$$

Harsanyi type structure for G (exogenous interactive beliefs):

$$\bar{\mathcal{T}} = \left(\Theta_i, \bar{T}_i, \bar{\beta}_i : \bar{T}_i \to \Delta\left(\Theta_{-i} \times \bar{T}_{-i}\right)\right)_{i \in I}$$

Bayesian game based on G: $\left(G, \bar{\mathcal{T}}\right)$ (as many as the Harsanyi type structures based on G)

Definition

A profile of maps $\sigma = (\sigma_i : \Theta_i \to S_i)_{i \in I}$ is **consistent with Bayesian equilibrium** if there are $\bar{\mathcal{T}}$ and $\bar{\sigma} = \left(\bar{\sigma}_i : \Theta_i \times \bar{T}_i \to S_i\right)_{i \in I}$ such that
(1) $\bar{\sigma}$ is a Bayesian equilibrium of $\left(G, \bar{\mathcal{T}}\right)$ and
(2) for every $\theta \in \Theta$ there is some $\bar{t} \in \bar{T}$ so that $\sigma(\theta) = \bar{\sigma}(\theta, \bar{t})$.

EGT and Bayesian Games (2)

Robust equilibrium predictions

From B&S BEJTE 2003 (cf. Brandenbuger & Dekel ECTA 1987):

Theorem

Fix a **static** *game with payoff uncertainty G. For every $\sigma \in \times_{i \in I} S_i^{\Theta_i}$, σ is consistent with Bayesian equilibrium* **if and only if** *it is consistent with RCBR, that is, $\sigma_i(\theta_i) \in \bigcap_{k>0} S_i^k(\theta_i)$ (rationalizable) for every $i \in I$ and $\theta_i \in \Theta_i$.*

Comment. The fully robust predictions of Bayesian equilibrium (i.e., independent of assumptions about exogenous interactive beliefs) are characterized by consistency with RCBR. Thus, fully robust implementation with static mechanisms must be implementation w.r.t. ("belief free") rationalizability.
We will see that using dynamic mechanisms and relying on forward-induction reasoning changes the picture.

Overview — First-Price Auctions

Battigalli & Siniscalchi, GEB 2003; Kosenkova, mimeo 2019

- Motivation: experiments challenging RNNE predictions
- Want to suggest strategic uncertainty may explain evidence
- Strategies (actions) are bids: $S_i = [0, M]$ for some large $M > 1$
- Today: 2 bidders, IPV on $[0, 1]$; $\Theta_i = [0, 1]$, valuation
- Today: beliefs as (deterministic) bid functions $m : \Theta_{-i} \to S_{-i}$
- Δ-RCBR; characterized by $S_i^{\Delta,k}(\theta_i)$

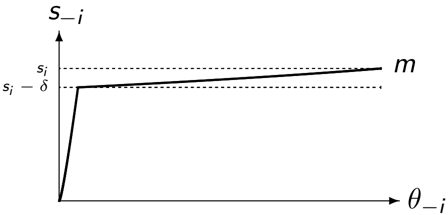
First-Price Auctions Example: round $k=1$

"Belief-free" RCBR (no Δ-restrictions) makes no prediction:
Any $s_i \in [0, M)$ is a BR to belief $m(\theta_{-i}) \equiv M$

Thus, Δ-restrictions: positive bids win with positive probability
Formally: beliefs $m(\cdot)$ continuous, strictly increasing, and $m(0) = 0$
Also, uniform distribution of valuations.

Claim: for $\theta_i > 0$, $S_i^{\Delta,1}(\theta_i) = (0, \theta_i)$

- $s_i \geq \theta_i$: win w/positive prob, get at most 0; deviate to $(0, \theta_i)$
- For $s_i \in (0, \theta_i)$:



First-Price Auctions Example: round $k = 2$ (1)

To sum up: $S_i^{\Delta,1}(\theta_i) = (0, \theta_i)$ for $\theta_i > 0$

Define $m^1(\theta_{-i}) \equiv \theta_{-i}$: upper bound of $S_{-i}^{\Delta,1}(\cdot)$

Now consider s_i with $0 < \theta_i - s_i < \max_{s_i'} \mathrm{E}u_i(\theta_i, s_i', m^1(\theta_{-i}))$

- Belief $m \in \Delta_i$ for round $k = 2$: $0 < m(\theta_{-i}) < \theta_{-i} \equiv m^1(\theta_{-i})$
- $m^1(\cdot)$ is the most pessimistic belief at round $k = 2$
- Let $\bar{s}_i = \arg\max_{s_i'} \mathrm{E}u_i(\theta_i, s_i', m^1(\theta_{-i}))$
- Surely $\mathrm{E}u_i(\theta_i, s_i, m(\theta_{-i})) \leq \theta_i - s_i$
- By assumption $\theta_i - s_i < \mathrm{E}u_i(\theta_i, \bar{s}_i, m^1(\theta_{-i}))$
- By the choice of m, $\mathrm{E}u_i(\theta_i, \bar{s}_i, m^1(\theta_{-i})) \leq \mathrm{E}u_i(\theta_i, \bar{s}_i, m(\theta_{-i}))$
- Conclude that $\mathrm{E}u_i(\theta_i, s_i, m(\theta_{-i})) < \mathrm{E}u_i(\theta_i, \bar{s}_i, m(\theta_{-i}))$

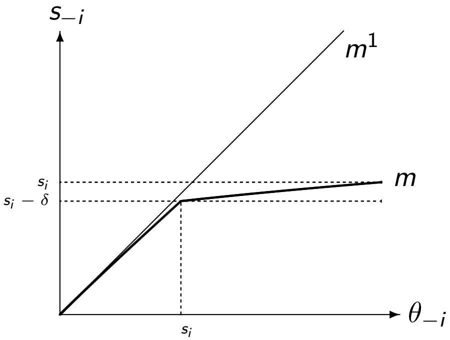
Thus, $s_i \notin S_i^{\Delta,2}(\theta_i)$!

First-Price Auctions Example: round $k = 2$ (2)

Let $m^2(\theta_i) = \theta_i - \max_{s_i'} \mathrm{E} u_i(\theta_i, s_i', m^1(\theta_{-i}))$

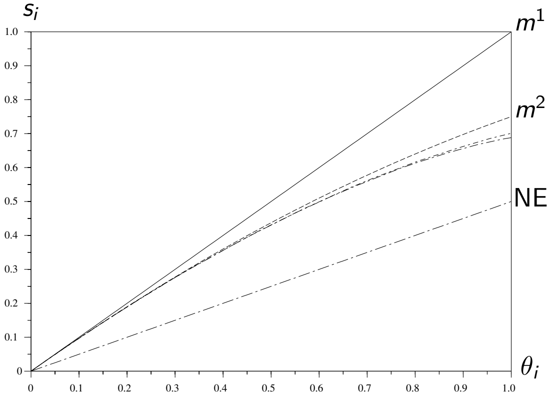
We showed $S_i^{\Delta,2}(\theta_i) \subseteq (0, m^2(\theta_i))$

Claim: bound is **tight**. Suppose $\theta_i - s_i > \max_{s_i'} \mathrm{E} u_i(\theta_i, s_i', m^1(\theta_{-i}))$



Thus, $BR_i(\theta_i, m) = \{s_i\}$. And can adapt for all k!

First-Price Auctions Example: some bounds



- Heterogeneity of bids
- Overbidding relative to RNNE
- Decreasing proportional deviations from RNNE

Related: Dekel & Wolinsky 2003 (fixed discrete grid of bids, increasing no. bidders approximate competitive bids)

Taking it to the data!

Kosenkova (2019): Non-parametric inference in first-price auctions with k-rationalizable beliefs

- k-rationalizability: θ_i bids below $m^k(\theta_i)$
- Let $G(\cdot)$ be CDF of bids in FP auction: observed
- Let $F_0(\cdot)$ be CDF of θ_i: separately identified from related ascending auction (application: timber tracts)
- Then can test for k-rationalizability:

$$\forall s_i, \qquad F_0\left((m^k)^{-1}(s_i)\right) \leq G(s_i)$$

Evidence of heterogeneous beliefs, so strategic uncertainty!

Overview - Robust Implementation

Bergemann & Morris REStud 2009, Robust implementation in direct mechanisms (plus earlier ECMA and later TE papers)

- Full implementation, Incomplete Information
- Beyond single Harsanyi type structures: Wilson Doctrine
- Use "belief-free" RCBR (via "belief-free" rationalizability)
- Characterize implementable SCF
- Later: RCSBR, strong rationalizability

Setup

- Outcomes X, finite or compact; allow lotteries $Y = \Delta(X)$
- Agent i has vNM utility $u_i : \Theta \times Y \to \mathbb{R}$
- Social choice **function**: $f : \Theta \to Y$
- Mechanism: $\mathcal{M} = (M_1, \ldots, M_N, g)$, with $g : M \to Y$

Given a mechanism $\mathcal{M}$, "belief-free" rationalizability: $S_i^k(\theta_i)$, $S_i^{\mathcal{M}}(\theta_i) = \bigcap_{k>0} S_i^k(\theta_i)$

Definition

SCF f is **robustly implemented** using $\mathcal{M}$ if, for every $\theta \in \Theta$ and $m \in M$, $m \in S^{\mathcal{M}}(\theta) \Rightarrow g(m) = f(\theta)$

Example: Public Good (1)

$X = X_0 \times \prod_i X_i$: amount of good, transfers.

$\Theta_i = [0, 1]$: agent's valuation type. But, interdependence:

$$u_i(\theta, x) = (\theta_i + \gamma \sum_{j \neq i} \theta_j) x_0 + x_i$$

Cost: $c(x_0) = \frac{1}{2} x_0^2$.

Efficient provision of x_0:

$$f_0(\theta) = (1 + \gamma(N - 1)) \sum_i \theta_i$$

Example: Public Good (2)

Will study **direct mechanism**: $S_i = \Theta_i$

Transfers:

$$f_i(\theta) = -(1 + \gamma(N-1)) \left(\gamma \theta_i \sum_{j \neq i} \theta_j + \frac{1}{2} \theta_i^2 \right)$$

Can show that, at (θ_i, θ_{-i}), given s_{-i}, best reply is

$$s_i = \theta_i + \gamma \sum_{j \neq i} (\theta_j - s_j)$$

- "Correct" for others' misreport
- Ex-post incentive compatibility: if $s_{-i} = \theta_{-i}$, report truth
- ...so above transfers are EPIC (though not DS) regardless of γ

Example: Public Good (3)

Now construct rationalizable reports $S_i^k(\theta_i) = [\underline{S}_i^k(\theta_i), \overline{S}_i^k(\theta_i)]$.

Due to linear-quadratic payoffs, enough to consider point beliefs: payoff type θ_{-i}, message $s_{-i} = \theta'_{-i}$.

Proposition

Assume $\gamma < \frac{1}{N-1}$. *Then, for every* θ_i,
$\underline{S}_i^k(\theta_i) = \max\{0, \theta_i - (\gamma(N-1))^k\}$ *and*
$\overline{S}_i^k(\theta_i) = \min\{1, \theta_i + (\gamma(N-1))^k\}$ ▸ *Proof*

So, for $\gamma < \frac{1}{N-1}$, $S_i^{\mathcal{M}}(\theta_i) = \{\theta_i\}$. **Full implementation!** (in DM)

Example: Public Good (4)

Suppose instead $\gamma > \frac{1}{N-1}$. Consider **arbitrary mechanism**

For each θ_i, let $\theta_j[\theta_i] \equiv \frac{1}{2} + \frac{1}{\gamma(N-1)}\left(\frac{1}{2} - \theta_i\right)$: in $[0,1]$

"Belief-free" RCBR allows this: any θ_i can have any belief about Θ_{-i}

But then

$$u_i\left(\theta_i, \theta_{-i}[\theta_i], x\right) = \left(\theta_i + \gamma \sum_{j \neq i} \theta_j[\theta_i]\right) x_0 + x_i = \frac{1}{2}(1+\gamma(N-1))x_0 + x_i$$

independent of θ_i.

Given these beliefs about Θ_{-i}, for each k and any belief about S_{-i}, any two types θ_i, θ_i' have the same BR's: **no implementation**

Key: with $\gamma > \frac{1}{N-1}$, θ_i's are **strategically indistinguishable**

Strategic distinguishability

Bergemann & Morris TE 2009: robust virtual implementation

Definition

Types $\theta_i, \theta_i' \in \Theta_i$ are strategically indistinguishable iff, for every mechanism $\mathcal{M}$, $S_i^{\mathcal{M}}(\theta_i) \cap S_i^{\mathcal{M}}(\theta_i') \neq \emptyset$

In every $\mathcal{M}$ there is an s_i that can be played by both θ_i or θ_i'

Direct characterization; Relate to Abreu & Matsushima, mimeo 1992

Definition

SCF f is **robustly measurable** if, for every i, θ_i, θ_i' and θ_{-i}, if types θ_i, θ_i' are strategically indistinguishable, then $f(\theta_i, \theta_{-i}) = f(\theta_i', \theta_{-i})$

BM: robust measurability + EPIC "iff" robust virtual impl.

But Example: robust measurability fails for $\gamma > \frac{1}{N-1}$

Implementation in Dynamic Mechanisms

Christoph Mueller JET 2016: Robust Virtual Implementation under Common Strong Belief in Rationality

- To recap, robust measurability is restrictive
- In turn, this is due to strategic distinguishability
- Key insight: forward-induction reasoning can separate more types in dynamic mechanisms!
- Hence, can weaken measurability
- Generic Robust Virtual Implementation under only EPIC in suitable class of environments!

Dynamic Epistemic Games

- Today: multistage game with observed actions
- Histories $\mathcal{H}$
- CPS: beliefs at every history related *via* Chain Rule $\Delta^{\mathcal{H}}(\Theta_{-i} \times S_{-i} \times T_{-i})$
- Local belief at h:
  $\beta_{i,h}(t_i) \in \Delta(\Theta_{-i} \times S_{-i}(h) \times T_{-i})$,
  $S_{-i}(h) =$ strategies that allow h

Rationality and Beliefs in Dynamic Games

R_i is now sequential rationality

Local belief at h:

$$\mathrm{B}_{i,h}(E_{-i}) = \{(\theta_i, s_i, t_i) : \beta_{i,h}(t_i)(E_{-i}) = 1\}$$

Strong belief (Battigalli & Siniscalchi JET 2002):

$$\mathrm{SB}_i(E_{-i}) = \bigcap_{h: E_{-i} \cap [\Theta_{-i} \times S_{-i}(h) \times T_{-i}] \neq \emptyset} \mathrm{B}_{i,h}(E_{-i})$$

Belief whenever possible

RCSBR and Strong Rationalizability

RCSBR – Rationality and common strong belief in rationality:

$$R_i^1 = R_i, \qquad R_i^k = R_i^{k-1} \cap \mathrm{SB}_i(R_{-i}^{k-1}) \quad (k > 1)$$

- Captures forward induction
- Characterization (rich type structures): strong rationalizability
- (Δ-restrictions also possible; not today)

Indistinguishability: static vs. dynamic (1)

u_1, u_2	w	x	y	z
θ_1, θ_2	5,1	1,0	0,3	2,2
θ_1, θ_2'	1,0	1,1	0,3	0,2
θ_1', θ_2	1,0	0,1	2,3	0,2
θ_1', θ_2'	5,0	1,1	0,0	2,1

- Fix $\mathcal{M} = (M_1, M_2, g)$ and $m_i^0 \in M_i$, $i = 1, 2$
- Choose $m_1^1 \in BR_1(\theta_1, \delta_{(\theta_2, m_2^0)})$.
  Since $u_1(\theta_1, \theta_2, \cdot) = u_1(\theta_1', \theta_2', \cdot)$, also $m_1^1 \in BR_1(\theta_1', \delta_{(\theta_2', m_2^0)})$
- Similarly, there is $m_2^1 \in BR_2(\theta_2, \delta_{(\theta_1', m_1^0)}) \cap BR_2(\theta_2', \delta_{(\theta_1, m_1^0)})$
- Inductively, find $m_1^k \in BR_1(\theta_1, \delta_{(\theta_2, m_2^{k-1})}) \cap BR_1(\theta_1', \delta_{(\theta_2', m_2^{k-1})})$
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- Finite game: stop at some K, so $m_i^K \in S_i^{\mathcal{M}}(\theta_i) \cap S_i^{\mathcal{M}}(\theta_i')$
- Note: same argument holds for dynamic mech with RICBR

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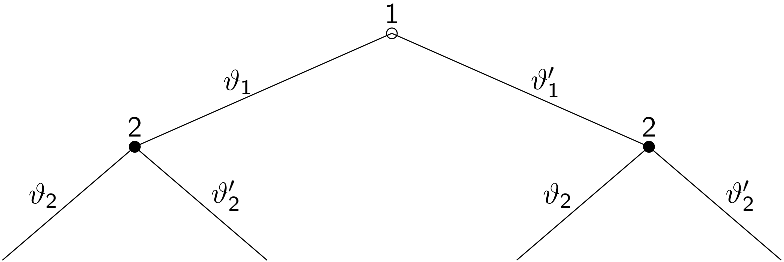
- Note: same argument holds for dynamic mech with RICBR

Indistinguishability: static vs. dynamic (2)

Now consider RCSBR / Strong Rationalizability in the this game



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θ_1, θ_2'	1,0
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θ_1', θ_2'	5,0

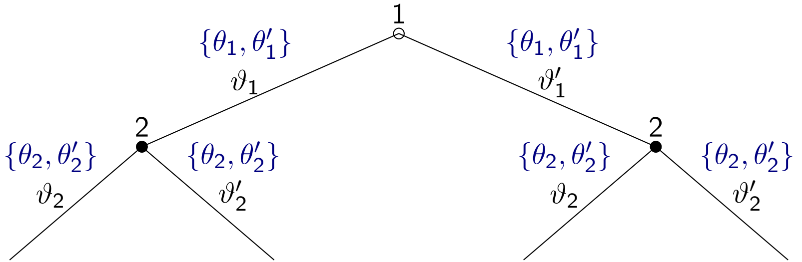
	x
θ_1, θ_2	1,0
θ_1, θ_2'	1,1
θ_1', θ_2	0,1
θ_1', θ_2'	1,1

	y
θ_1, θ_2	0,3
θ_1, θ_2'	0,3
θ_1', θ_2	2,3
θ_1', θ_2'	0,0

	z
θ_1, θ_2	2,2
θ_1, θ_2'	0,2
θ_1', θ_2	0,2
θ_1', θ_2'	2,1

Indistinguishability: static vs. dynamic (2)

Now consider RCSBR / Strong Rationalizability in the this game



	w
θ_1, θ_2	5,1
θ_1, θ_2'	1,0
θ_1', θ_2	1,0
θ_1', θ_2'	5,0

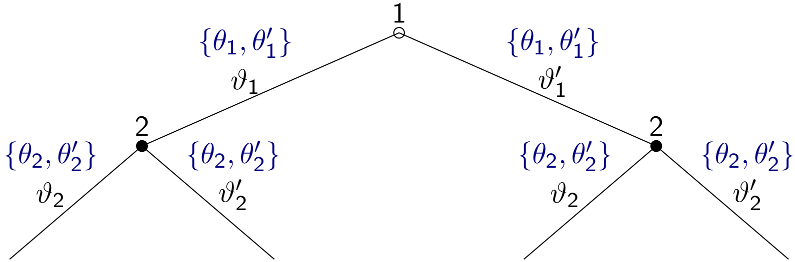
	x
θ_1, θ_2	1,0
θ_1, θ_2'	1,1
θ_1', θ_2	0,1
θ_1', θ_2'	1,1

	y
θ_1, θ_2	0,3
θ_1, θ_2'	0,3
θ_1', θ_2	2,3
θ_1', θ_2'	0,0

	z
θ_1, θ_2	2,2
θ_1, θ_2'	0,2
θ_1', θ_2	0,2
θ_1', θ_2'	2,1

Indistinguishability: static vs. dynamic (2)

Now consider RCSBR / Strong Rationalizability in the this game



	w
θ_1, θ_2	5,1
θ_1, θ_2'	1,0
θ_1', θ_2	1,0
θ_1', θ_2'	5,0

	x
θ_1, θ_2	1,0
θ_1, θ_2'	1,1
θ_1', θ_2	0,1
θ_1', θ_2'	1,1

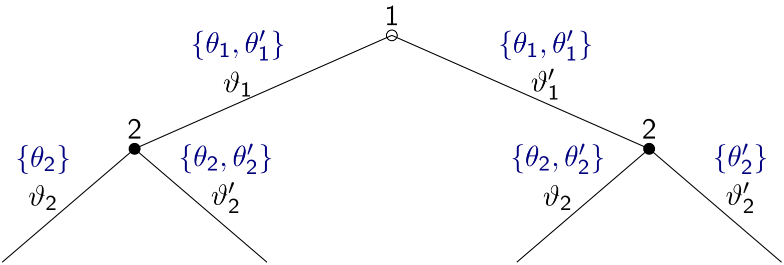
	y
θ_1, θ_2	0,3
θ_1, θ_2'	0,3
θ_1', θ_2	2,3
θ_1', θ_2'	0,0

	z
θ_1, θ_2	2,2
θ_1, θ_2'	0,2
θ_1', θ_2	0,2
θ_1', θ_2'	2,1

$k = 1$ after ϑ_1', θ_2 will not play ϑ_2';
after ϑ_1, θ_2' will not play ϑ_2

Indistinguishability: static vs. dynamic (2)

Now consider RCSBR / Strong Rationalizability in the this game



	w
θ_1, θ_2	5,1
θ_1, θ'_2	1,0
θ'_1, θ_2	1,0
θ'_1, θ'_2	5,0

	x
θ_1, θ_2	1,0
θ_1, θ'_2	1,1
θ'_1, θ_2	0,1
θ'_1, θ'_2	1,1

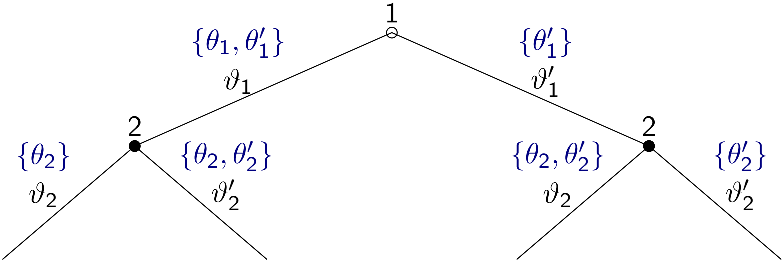
	y
θ_1, θ_2	0,3
θ_1, θ'_2	0,3
θ'_1, θ_2	2,3
θ'_1, θ'_2	0,0

	z
θ_1, θ_2	2,2
θ_1, θ'_2	0,2
θ'_1, θ_2	0,2
θ'_1, θ'_2	2,1

$k = 2$ θ_1 won't play ϑ'_1: if θ_2, ϑ'_1 yields y (worth 0), vs. w, x (5,1); and if θ'_2, ϑ'_1 yields y, z (0), vs. x (1). Thus $\theta_1 : \vartheta_1$

Indistinguishability: static vs. dynamic (2)

Now consider RCSBR / Strong Rationalizability in the this game



	w
θ_1, θ_2	5,1
θ_1, θ_2'	1,0
θ_1', θ_2	1,0
θ_1', θ_2'	5,0

	x
θ_1, θ_2	1,0
θ_1, θ_2'	1,1
θ_1', θ_2	0,1
θ_1', θ_2'	1,1

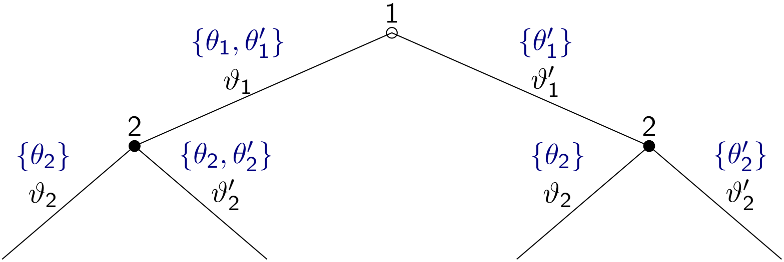
	y
θ_1, θ_2	0,3
θ_1, θ_2'	0,3
θ_1', θ_2	2,3
θ_1', θ_2'	0,0

	z
θ_1, θ_2	2,2
θ_1, θ_2'	0,2
θ_1', θ_2	0,2
θ_1', θ_2'	2,1

$k = 3$ after ϑ_1', θ_2' won't play ϑ_2: surely θ_1' (FI), and ϑ_2' yields z (1) whereas ϑ_2 yields y (0). Thus $\theta_2' : \vartheta_2'\vartheta_2'$

Indistinguishability: static vs. dynamic (2)

Now consider RCSBR / Strong Rationalizability in the this game



	w
θ_1, θ_2	5,1
θ_1, θ_2'	1,0
θ_1', θ_2	1,0
θ_1', θ_2'	5,0

	x
θ_1, θ_2	1,0
θ_1, θ_2'	1,1
θ_1', θ_2	0,1
θ_1', θ_2'	1,1

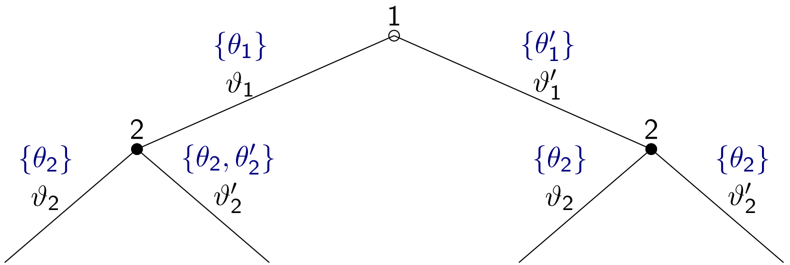
	y
θ_1, θ_2	0,3
θ_1, θ_2'	0,3
θ_1', θ_2	2,3
θ_1', θ_2'	0,0

	z
θ_1, θ_2	2,2
θ_1, θ_2'	0,2
θ_1', θ_2	0,2
θ_1', θ_2'	2,1

$k = 4$ θ_1' won't play ϑ_1: if θ_2, then ϑ_1 yields w, x (1,0) but ϑ_1' yields y (2); and if θ_2', then ϑ_1 yields x (1) vs. z (2). Thus $\theta_1' : \vartheta_1'$

Indistinguishability: static vs. dynamic (2)

Now consider RCSBR / Strong Rationalizability in the this game



	w
θ_1, θ_2	5,1
θ_1, θ'_2	1,0
θ'_1, θ_2	1,0
θ'_1, θ'_2	5,0

	x
θ_1, θ_2	1,0
θ_1, θ'_2	1,1
θ'_1, θ_2	0,1
θ'_1, θ'_2	1,1

	y
θ_1, θ_2	0,3
θ_1, θ'_2	0,3
θ'_1, θ_2	2,3
θ'_1, θ'_2	0,0

	z
θ_1, θ_2	2,2
θ_1, θ'_2	0,2
θ'_1, θ_2	0,2
θ'_1, θ'_2	2,1

$k = 5$ θ_2 won't play ϑ'_2 after ϑ_1: surely θ_1 (FI) and ϑ'_2 yields x (0) vs. w (1). Thus $\theta_2 : \vartheta_2\vartheta_2$. DONE

See Also

- Abreu & Matsushima, ECMA 1992; virtual (rationalizable) implementation under complete information
- Abreu & Matsushima, mimeo 1992; virtual implementation under incomplete information; measurability
- Artemov, Kunimoto & Serrano, JET 2013; robust virtual implementation with Δ-restrictions: Δ-IC, Δ-measurability
- Mueller, mimeo 2018: on robust implementation with "initial rationalizability" (RICBR)
- Penta, JET 2015; on robust implementation with "backward-induction rationalizability" (or something like that)
- Ollar & Penta, AER 2017; also Δ-rationalizable full implementation, but with specific "economic" belief restrictions
- Shimoji & Schweinzer GEB 2015; Δ-rationalizable implementation w/out IC, by partially informed planner

Conclusions

IT COULD WORK!

THANK YOU

BM Public Goods Example (a)

Initialize: $\underline{S}_i^0(\theta_i) = 0$, $\overline{S}_i^0(\theta_i) = 1$.

Recall $\theta_i' = \theta_i + \gamma \sum_{j \neq i}(\theta_j - \theta_j')$. At $k > 0$:

- Choose $(\theta_{-i}, \theta'_{-i})$ within $S_j^k(\cdot)$ to max / min each $(\theta_j - \theta_j')$
- To max, choose min $\theta_j' = \underline{S}_j^{k-1}(\theta_j)$
- To min, choose max $\theta_j' = \overline{S}_j^{k-1}(\theta_j)$
- Ensure we stay within $[0, 1]$

$$\underline{S}_i^k(\theta_i) = \max \left\{ 0, \theta_i + \gamma \min_{\theta_{-i}} \sum_{j \neq i} \left(\theta_j - \overline{S}_j^{k-1}(\theta_j) \right) \right\}$$

$$\overline{S}_i^k(\theta_i) = \min \left\{ 1, \theta_i + \gamma \max_{\theta_{-i}} \sum_{j \neq i} \left(\theta_j - \underline{S}_j^{k-1}(\theta_j) \right) \right\}$$

BM Public Goods Example (b)

Claim: for every θ_i,
$\underline{S}_i^k(\theta_i) = \max\{0, \theta_i - (\gamma(I-1))^k\}$, $\overline{S}_i^k(\theta_i) = \min\{1, \theta_i + (\gamma(I-1))^k\}$

Proof: true for $k = 1$ as $\underline{S}_j^0 \equiv 0$, $\overline{S}_j^0 \equiv 1$. So assume for $k - 1 \geq 0$.
For $\theta_j \leq 1 - [\gamma(N-1)]^{k-1}$, which is ≤ 1 by assumption, one has $\theta_j + [\gamma(I-1)]^{k-1} \leq 1$, so $\theta_j - \overline{S}_j^{k-1}(\theta_j) = -[\gamma(N-1)]^{k-1}$.
For $\theta_j > 1 - [\gamma(N-1)]^{k-1}$, $\theta_j - \overline{S}_j^{k-1}(\theta_j) = \theta_j - 1 > -[\gamma(N-1)]^{k-1}$.
Hence $\min_{\theta_j} \left[\theta_j - \overline{S}_j^{k-1}(\theta_j)\right] = -[\gamma(N-1)]^{k-1}$.
Thus
$\gamma \min_{\theta_{-i}} \sum_{j \neq i} \left(\theta_j - \overline{S}_j^{k-1}(\theta_j)\right) = \gamma(N-1) \cdot [-(\gamma(N-1))^{k-1}]$.
This implies the claim. Argument for other bound similar. □